Excited-State Energies and Scattering Phase-Shifts from Lattice QCD

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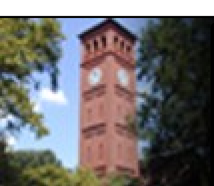
Time-separated Euclidean Correlator as Path-Integral on Lattice:

$$\left\langle \hat{O}^\dagger(t)\hat{O}(0) \right\rangle = \int [d\Phi]\, e^{-S[\Phi]} O\left[\Phi, t\right] O\left[\Phi, 0\right]$$

Time-separated Euclidean Correlator as Sum over Stationary-States:

$$\left\langle \hat{O}^\dagger(t)\hat{O}(0) \right\rangle = \sum_n |\left\langle 0 \right| \hat{O} \left| n \right\rangle|^2 e^{-E_n t}$$

- In principle, could find all energies... Not in practice.



Define Matrix of Correlators!

$$C_{ij}(t) = \left\langle \hat{O}_i^\dagger(t)\hat{O}_j(0) \right\rangle$$

$$= \sum_n \langle 0| \hat{O}_i |n\rangle \langle n| \hat{O}_j^\dagger |0\rangle \, e^{-E_n t}$$

Diagonalize to pick out Excited-State Energies:

$$\widetilde{C}_{ij}(t) = U_\tau^\dagger \, C_{ij}(t) \, U_\tau$$

- U diagonalizes C at $t = \tau$
- Choose τ such that C remains diagonal (approx.) for $t > \tau$

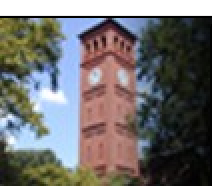
Correlator Eigenvalues:

$$\widetilde{C}_{nn}(t) = A_n e^{-E_n t} + \cdots$$

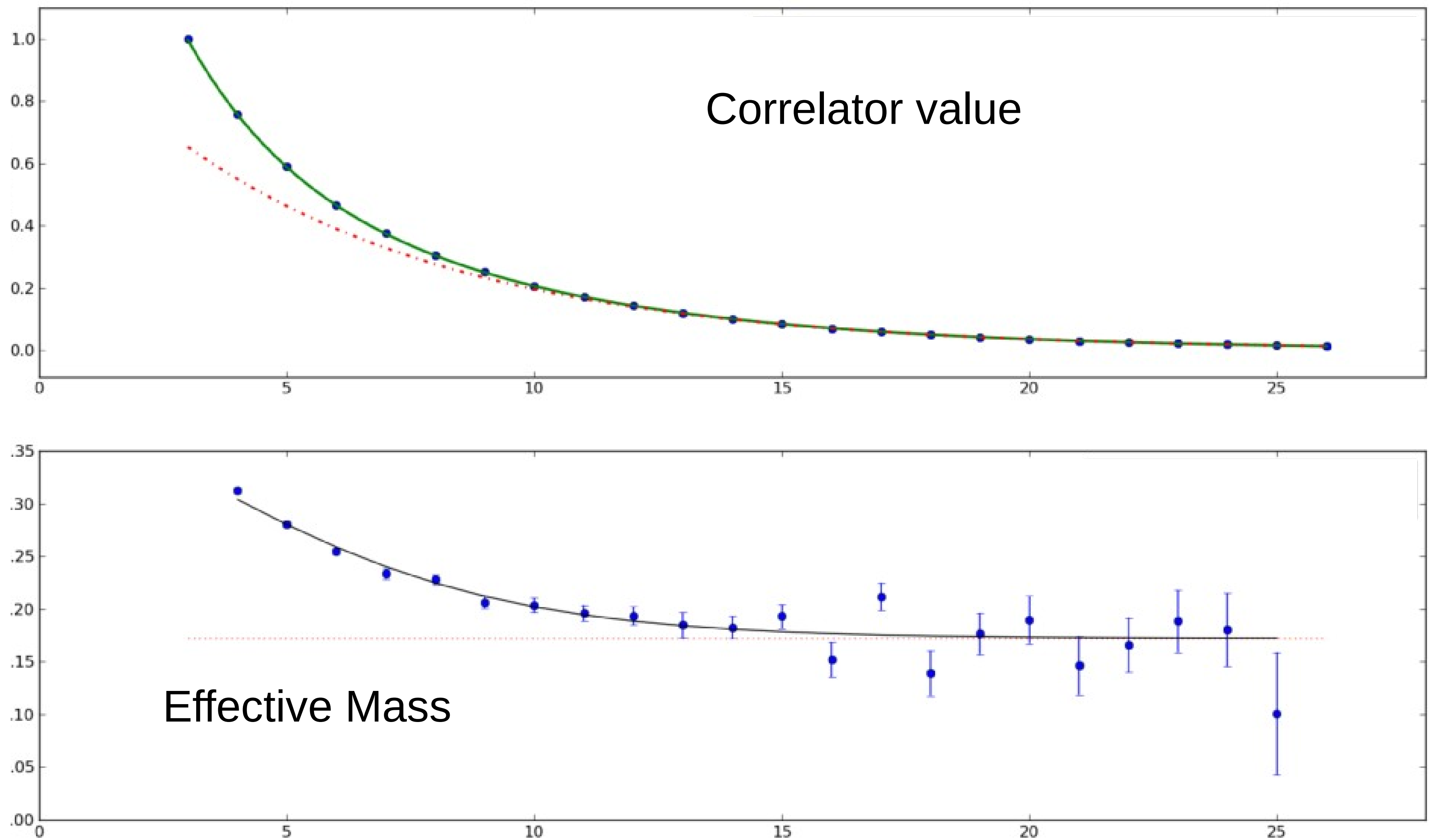
- Sum of decaying exponentials
- Only leading term contributes for large t → single exponential fit ?
- Capture excited state contamination → double exponential fit !

Effective Mass:

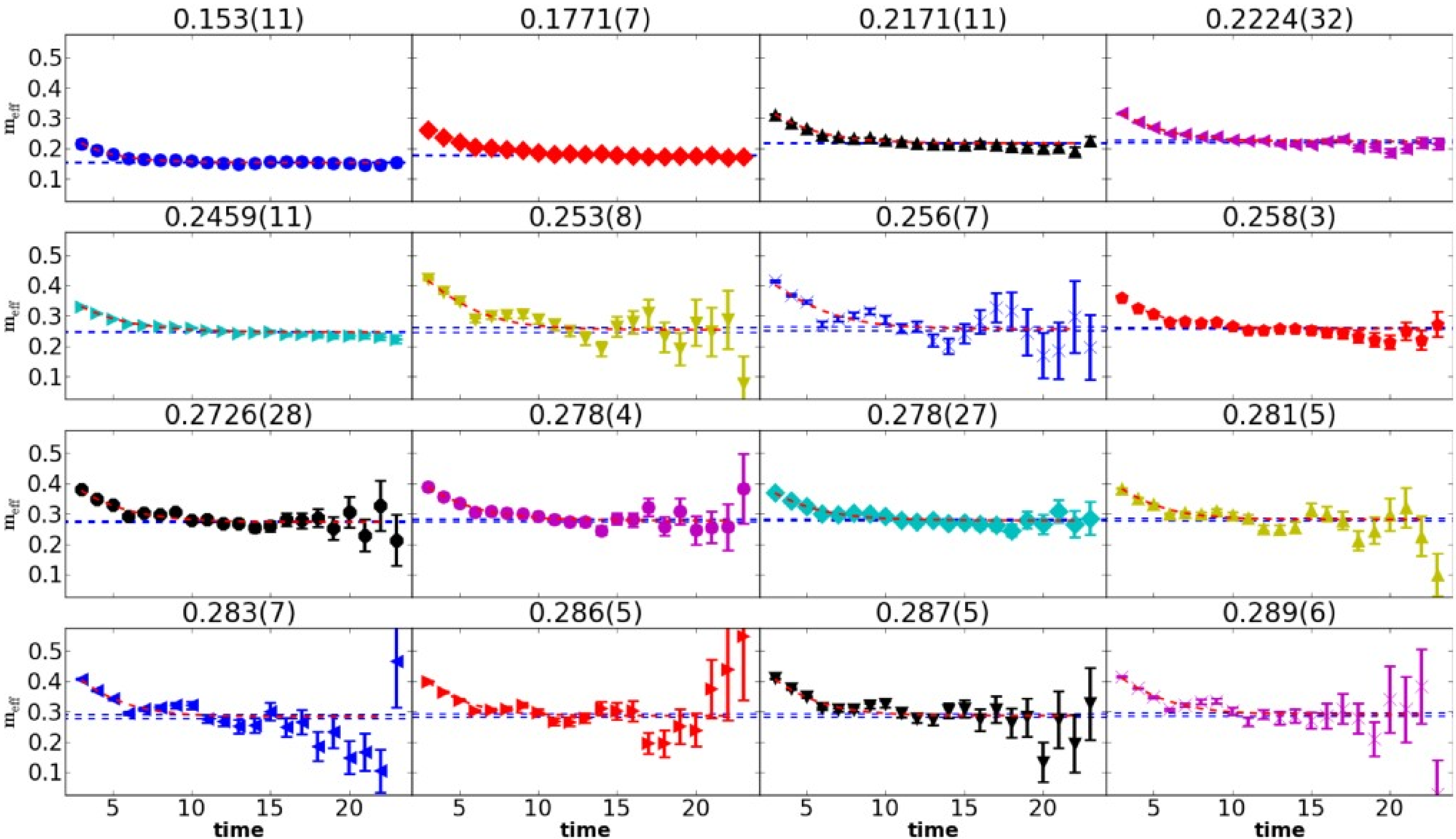
$$m_n^{\text{eff}}(t) = \frac{1}{\Delta t} \ln \left(\frac{\widetilde{C}_{nn}(t)}{\widetilde{C}_{nn}(t + \Delta t)} \right)$$



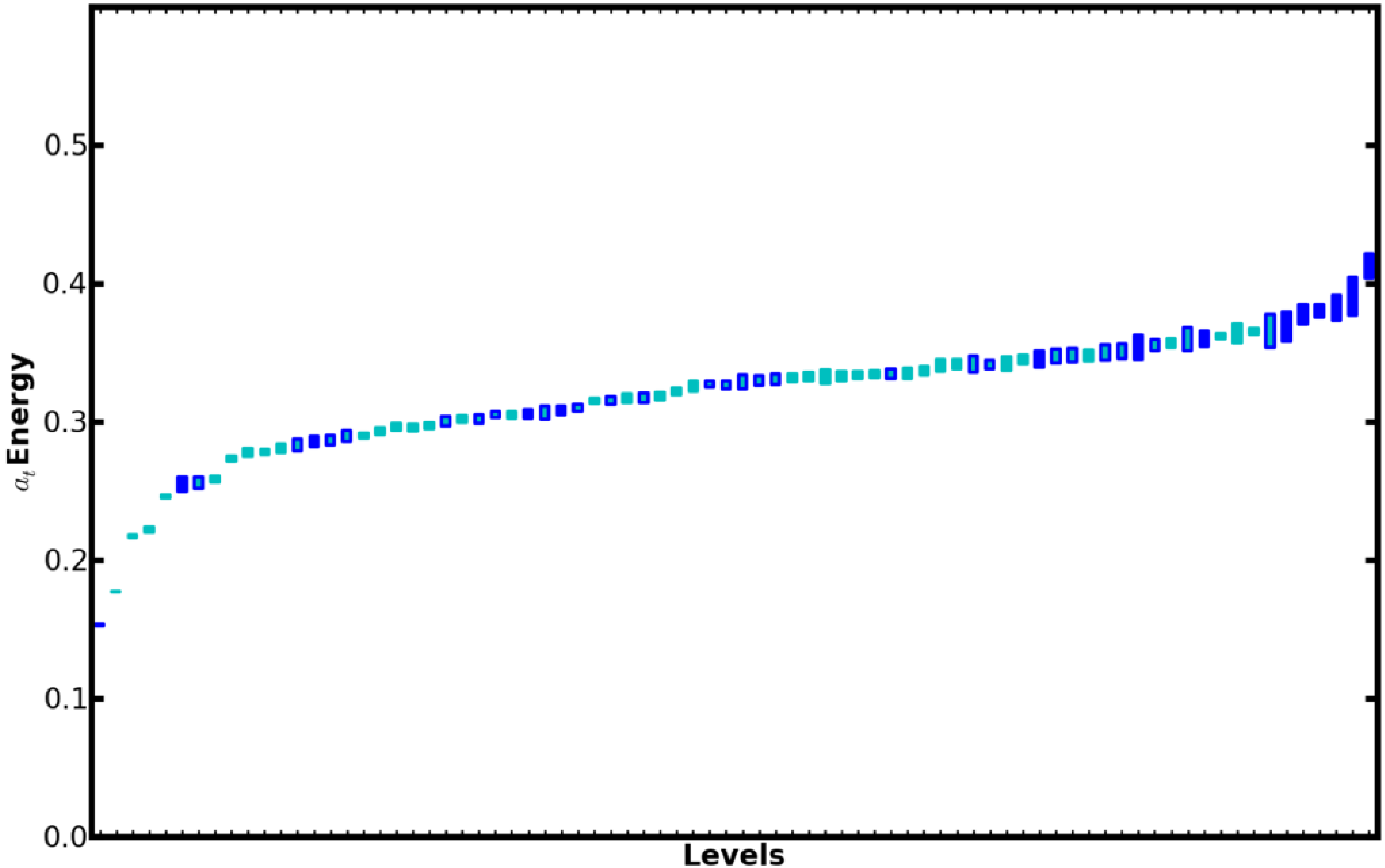
Sample Correlator from Isodoublet Strange T_{1u}:



More Effective Masses from Kaon Channel:



Stationary-State Energies in Kaon Channel:



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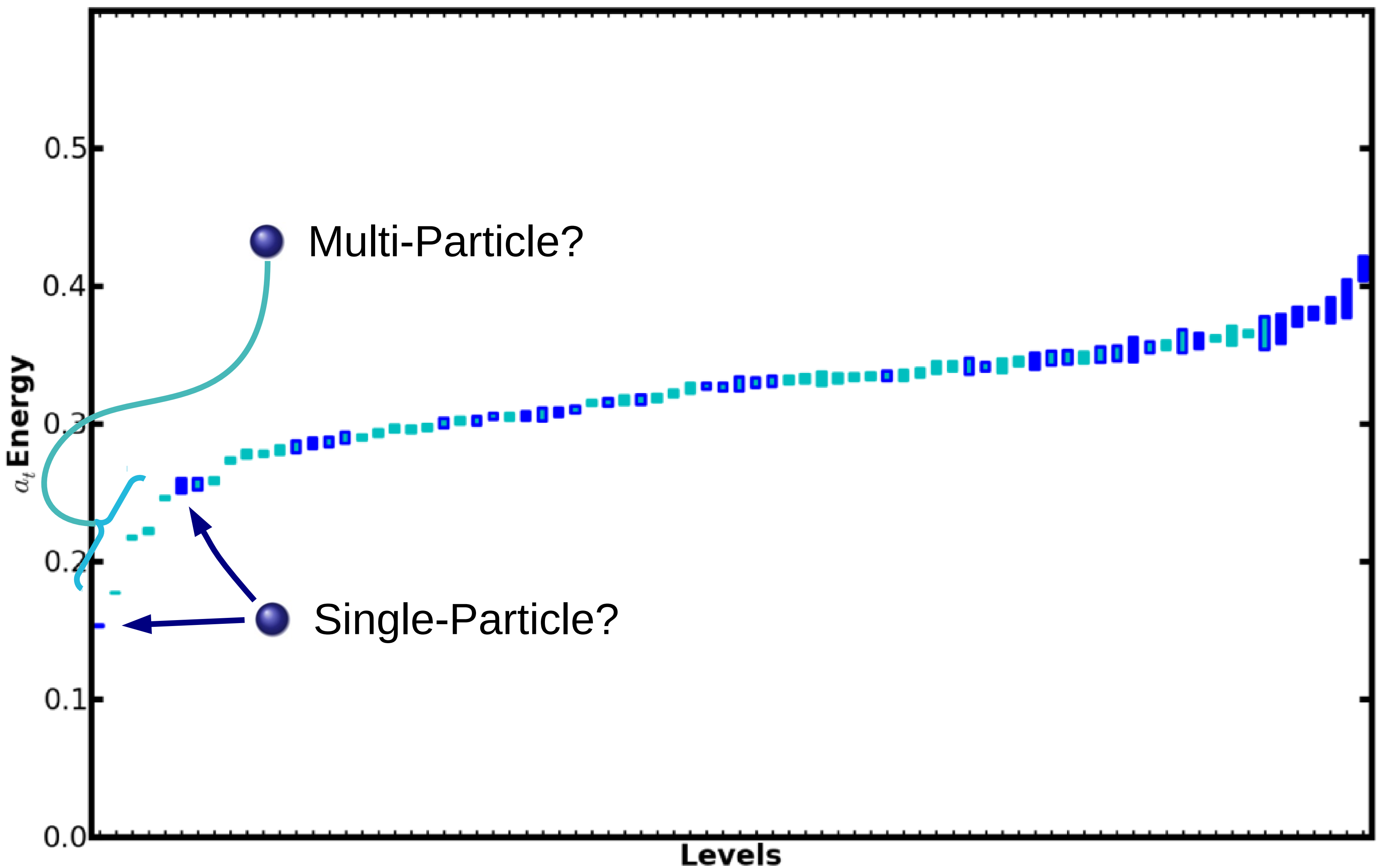
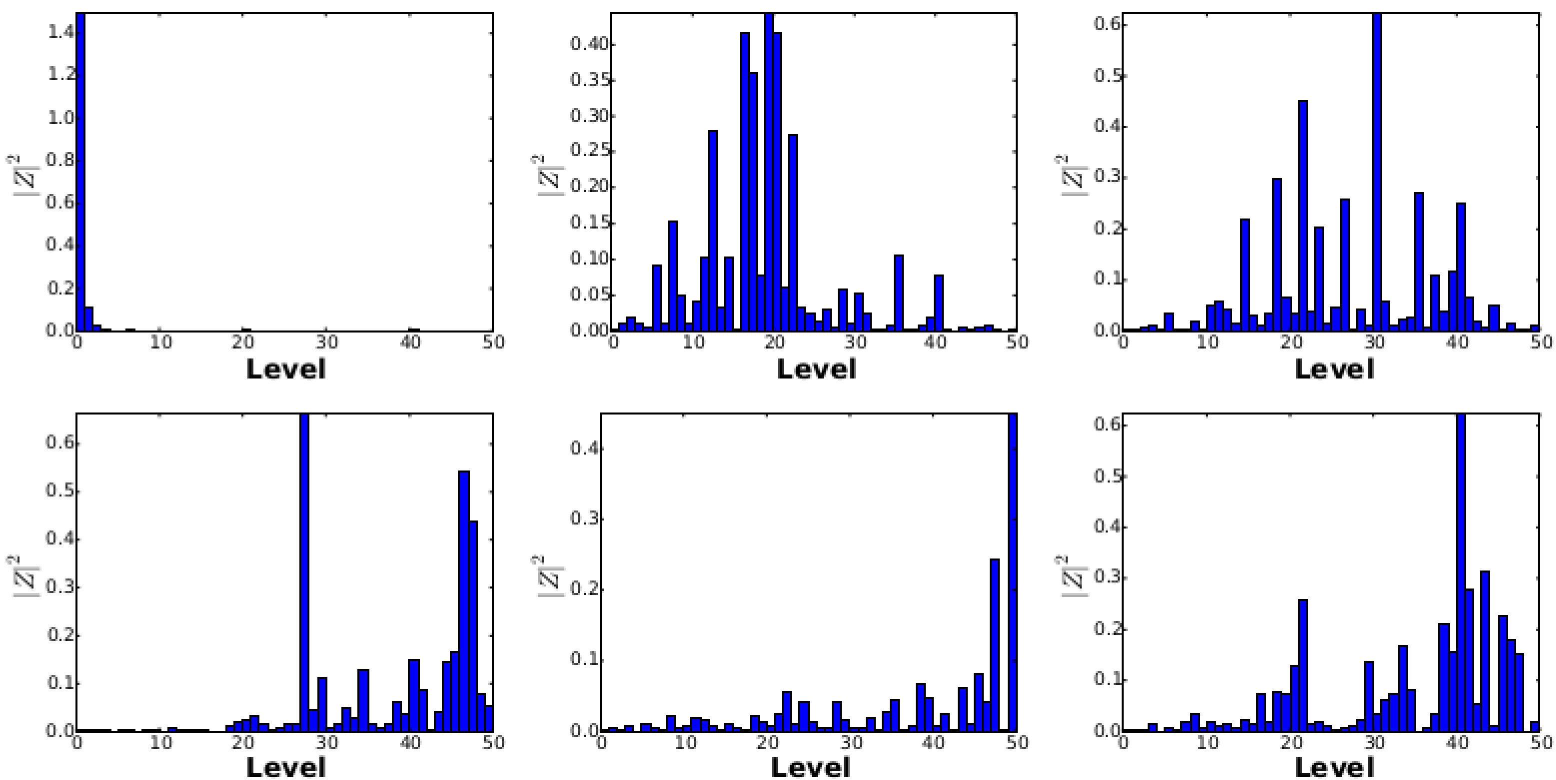


Identifying qqq, q$\bar{q}$ resonances

- Rebuild correlator matrix using ONLY single-hadron operators
- Diagonalize again to find appropriate linear combinations

$$\hat{O}_i'^{\mathrm{SH}} = \sum U'_{ij} \hat{O}_j^{\mathrm{SH}} \qquad \langle n | \, \hat{O}_i'^{\mathrm{SH}\dagger} \, | 0 \rangle = Z_i^{(n)}$$



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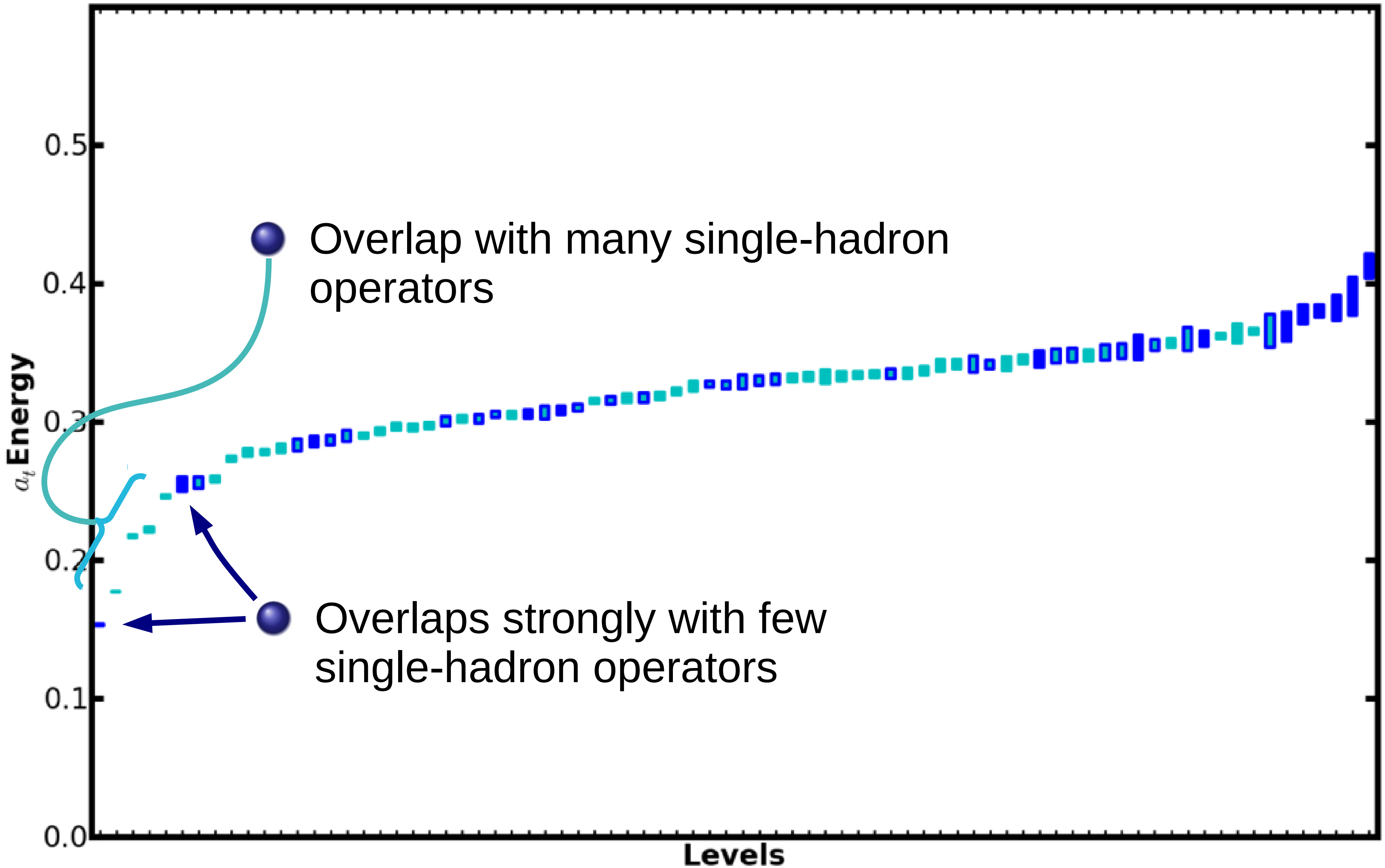
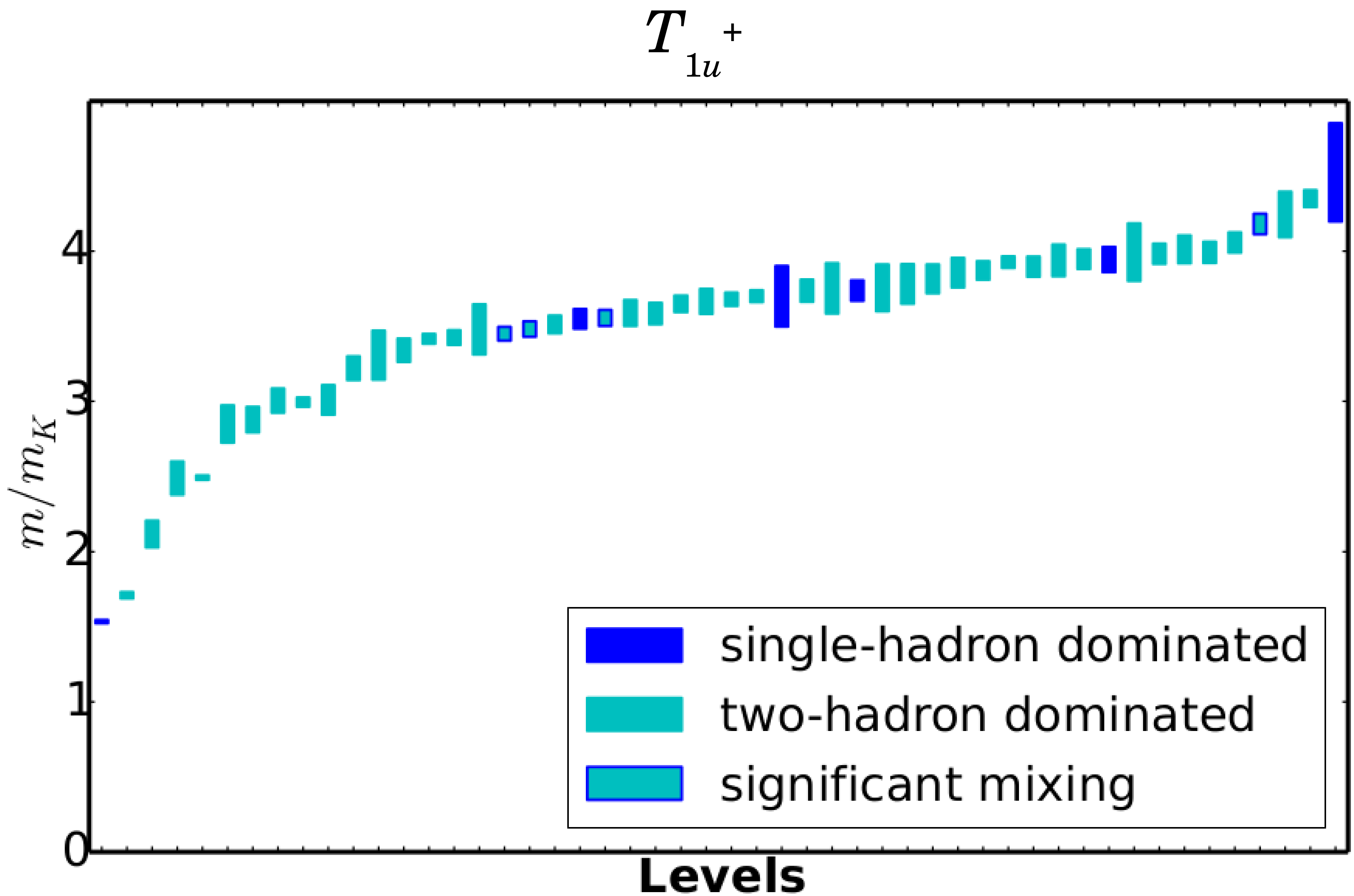


Stationary-State Energies in Rho Channel:



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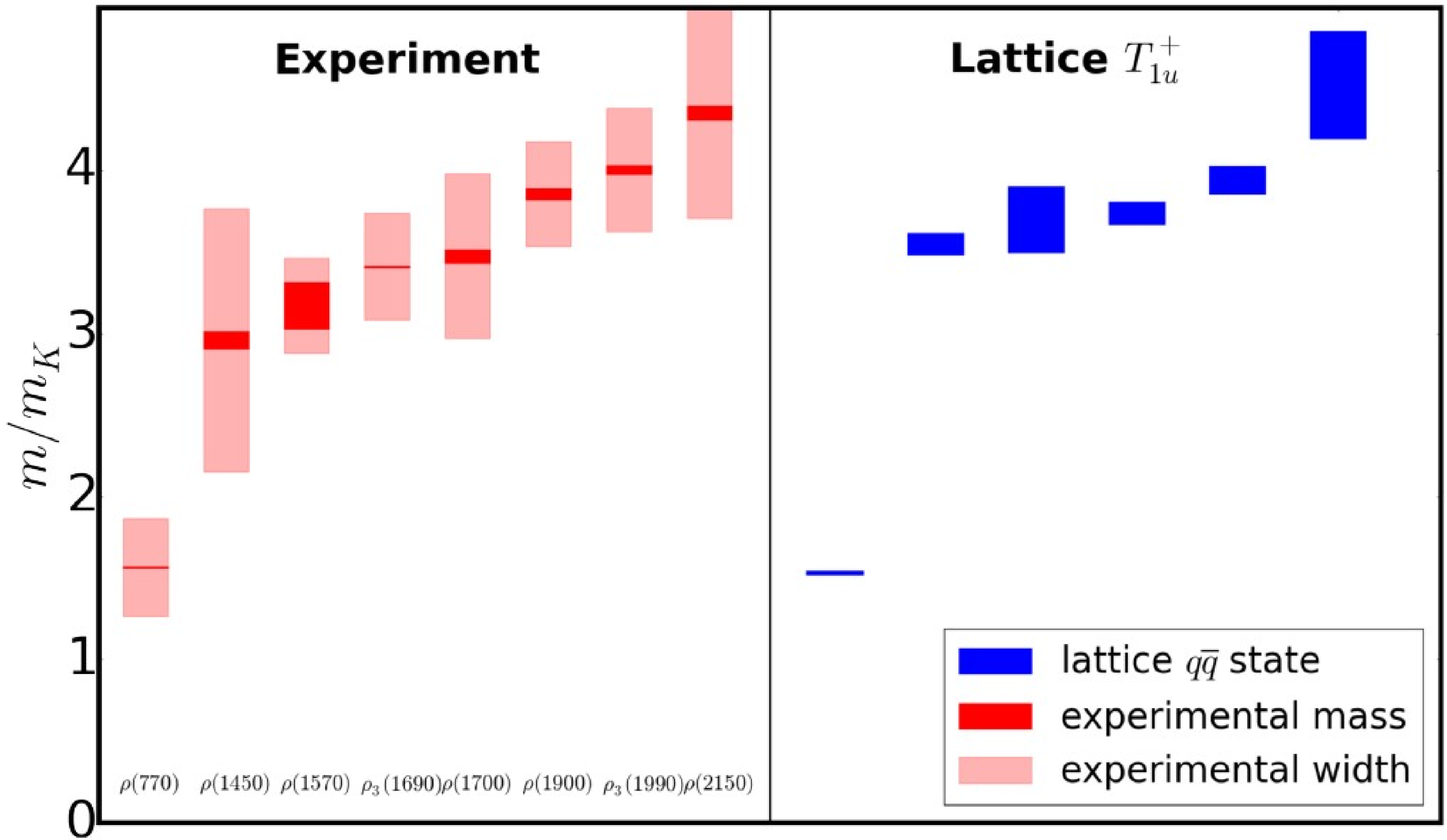
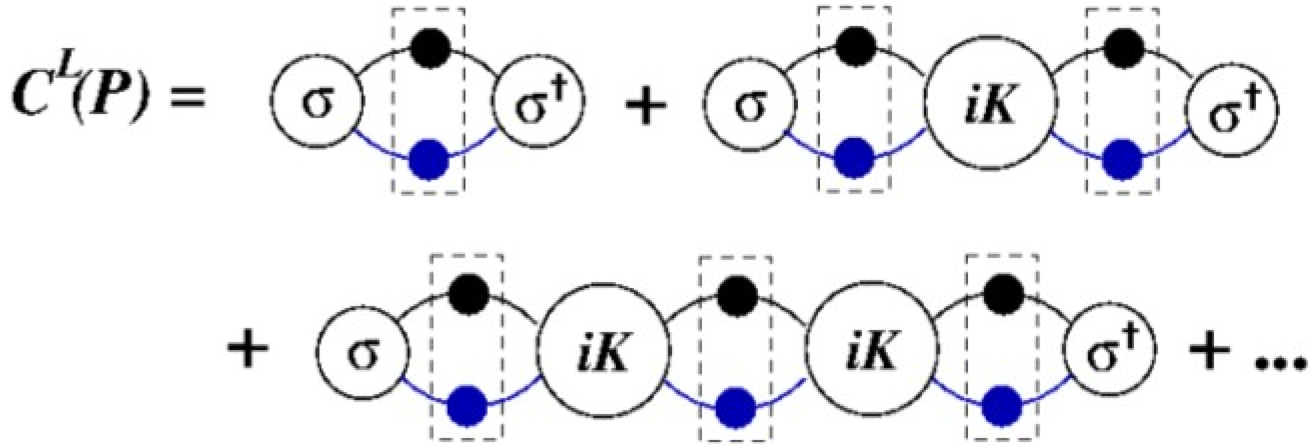
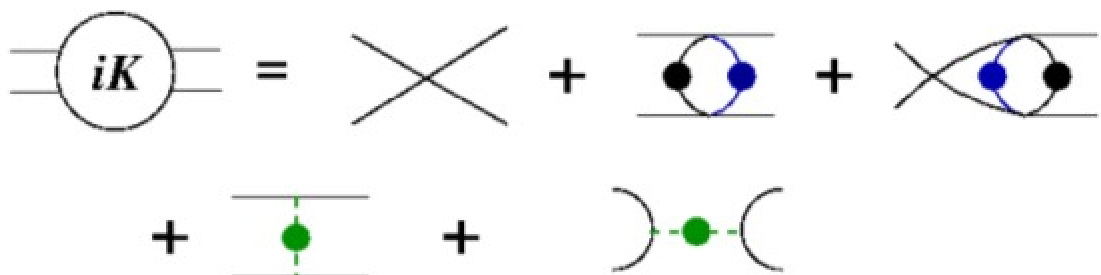


Scattering Phase-Shifts

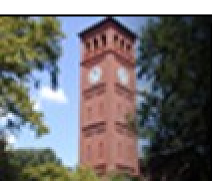
- correlator of two-particle operator σ in finite volume



- Bethe-Salpeter kernel

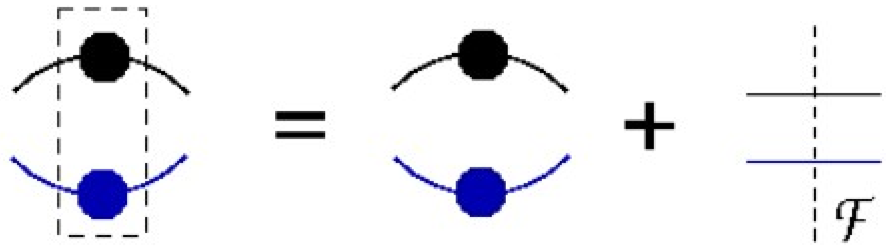


- $C^{\infty}(P)$ has branch cuts where two-particle thresholds begin
- momentum quantization in finite volume: cuts $\rightarrow$ series of poles
- C^{L} poles: two-particle energy spectrum of finite volume theory

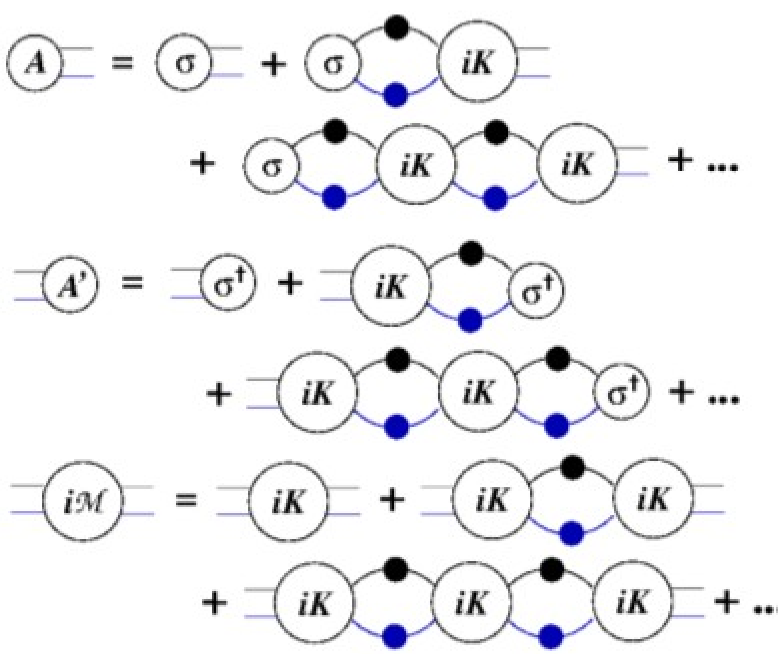


Finite-Volume as Infinite-Volume + Correction

- finite-volume momentum sum is infinite-volume integral plus correction $\mathcal{F}$

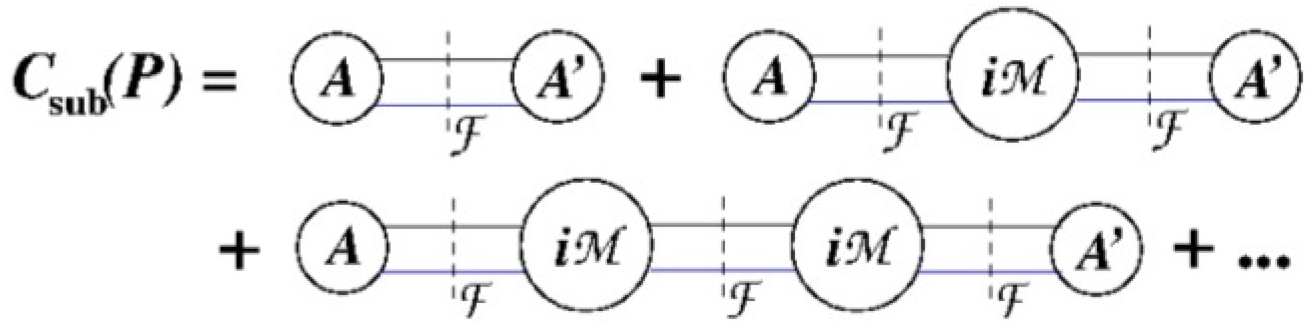


- define the following quantities: A, A', invariant scattering amplitude $i\mathcal{M}$



Difference Inherits Finite-Volume Poles

- subtracted correlator $C_{\rm sub}(P) = C^L(P) - C^\infty(P)$ given by



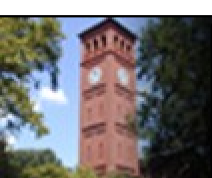
- sum geometric series

$$C_{\rm sub}(P) = A\,\mathcal{F}(1 - i\mathcal{M}\mathcal{F})^{-1}\,A'.$$

- poles of $C_{\rm sub}(P)$ are poles of $C^L(P)$ from $\det(1 - i\mathcal{M}\mathcal{F}) = 0$
- E related to S matrix (and phase shifts) by

$$\det[1 + F^{(\boldsymbol{s},\gamma,u)}(S - 1)] = 0,$$

where F matrix defined next slide



Too Many Functions

- F matrix in JLS basis states given by

$$F^{(\boldsymbol{s},\gamma,u)}_{J'm_{J'}L'S'a';\; Jm_JLSa} = \frac{\rho_a}{2}\delta_{a'a}\delta_{S'S}\Big\{\delta_{J'J}\delta_{m_{J'}m_J}\delta_{L'L}$$
$$+W^{(\boldsymbol{s},\gamma,u)}_{L'm_{L'};\; Lm_L}\langle J'm_{J'}|L'm_{L'},Sm_S\rangle\langle Lm_L,Sm_S|Jm_J\rangle\Big\}$$

$$W^{(\boldsymbol{s},\gamma,u)}_{L'm_{L'};\; Lm_L} = \frac{2i}{\pi\gamma u^{l+1}}\mathcal{Z}_{lm}(\boldsymbol{s},\gamma,u^2)\int d^2\Omega\, Y^*_{L'm_{L'}}(\Omega)Y^*_{lm}(\Omega)Y_{Lm_L}(\Omega)$$

$$\mathcal{Z}_{lm}(\boldsymbol{s},\gamma,u^2) = \sum_{\boldsymbol{n}\in\mathbb{Z}^3}\frac{\mathcal{Y}_{lm}(\boldsymbol{z})}{(\boldsymbol{z}^2-u^2)}e^{-\Lambda(\boldsymbol{z}^2-u^2)}$$
$$+\delta_{l0}\gamma\pi e^{\Lambda u^2}\left(2uD(u\sqrt{\Lambda})-\Lambda^{-1/2}\right)$$
$$+\frac{i^l\gamma}{\Lambda^{l+1/2}}\int_0^1 dt\left(\frac{\pi}{t}\right)^{l+3/2}e^{\Lambda t u^2}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^3\\ \boldsymbol{n}\neq 0}} e^{\pi i\boldsymbol{n}\cdot\boldsymbol{s}}\mathcal{Y}_{lm}(\mathbf{w})\,e^{-\pi^2\mathbf{w}^2/(t\Lambda)}$$

Too Many Functions (cont.)

$$\boldsymbol{z} = \boldsymbol{n} - \gamma^{-1}\left[\tfrac{1}{2} + (\gamma - 1)s^{-2}\boldsymbol{n}\cdot\boldsymbol{s}\right]\boldsymbol{s},$$

$$\mathbf{w} = \boldsymbol{n} - (1-\gamma)s^{-2}\boldsymbol{s}\cdot\boldsymbol{n}\boldsymbol{s}, \qquad \mathcal{Y}_{lm}(\mathbf{x}) = |\mathbf{x}|^l\, Y_{lm}(\widehat{\mathbf{x}})$$

$$D(x) = e^{-x^2}\int_0^x dt\; e^{t^2}$$ (Dawson function)

$$E_{\rm cm} = \sqrt{E^2 - \boldsymbol{P}^2}, \qquad \gamma = \frac{E}{E_{\rm cm}},$$

$$\boldsymbol{q}_{\rm cm}^2 = \frac{1}{4}E_{\rm cm}^2 - \frac{1}{2}(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4E_{\rm cm}^2},$$

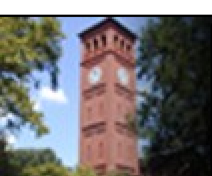
$$u^2 = \frac{L^2\boldsymbol{q}_{\rm cm}^2}{(2\pi)^2}, \qquad \boldsymbol{s} = \left(1 + \frac{(m_1^2 - m_2^2)}{E_{\rm cm}^2}\right)\boldsymbol{d}$$

Phase-Shifts in terms of Energies:

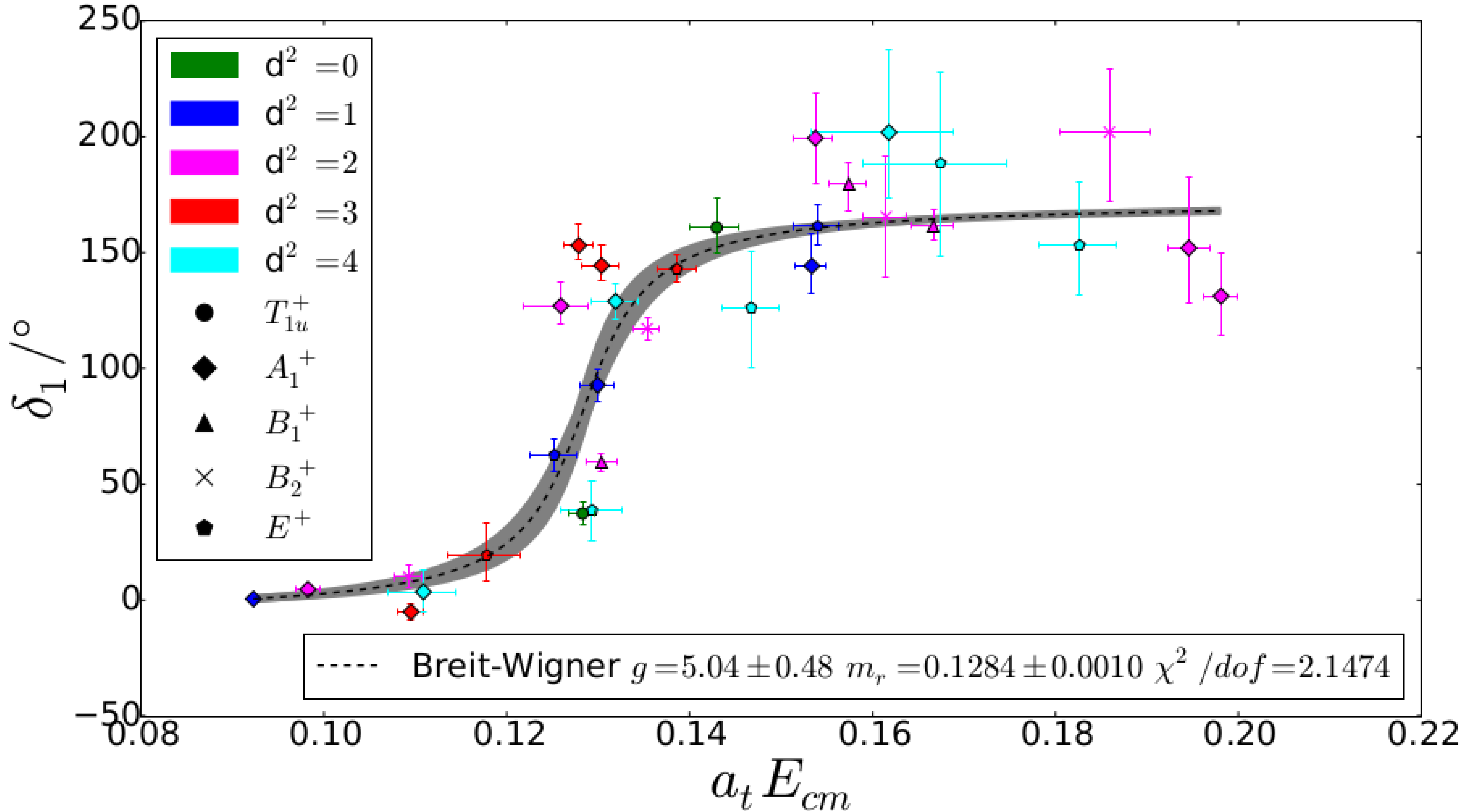
- for P-wave phase shift $\delta_1(E_{\text{cm}})$ for $\pi\pi$ $I=1$ scattering
- define

$$w_{lm} = \frac{\mathcal{Z}_{lm}(\boldsymbol{s}, \gamma, u^2)}{\gamma \pi^{3/2} u^{l+1}}$$

$\boldsymbol{d}$	Λ	$\cot \delta_1$
(0,0,0)	T_{1u}^+	$\text{Re } w_{0,0}$
(0,0,1)	A_1^+	$\text{Re } w_{0,0} + \frac{2}{\sqrt{5}} \text{Re } w_{2,0}$
	E^+	$\text{Re } w_{0,0} - \frac{1}{\sqrt{5}} \text{Re } w_{2,0}$
(0,1,1)	A_1^+	$\text{Re } w_{0,0} + \frac{1}{2\sqrt{5}} \text{Re } w_{2,0} - \sqrt{\frac{6}{5}} \text{Im } w_{2,1} - \sqrt{\frac{3}{10}} \text{Re } w_{2,2},$
	B_1^+	$\text{Re } w_{0,0} - \frac{1}{\sqrt{5}} \text{Re } w_{2,0} + \sqrt{\frac{6}{5}} \text{Re } w_{2,2},$
	B_2^+	$\text{Re } w_{0,0} + \frac{1}{2\sqrt{5}} \text{Re } w_{2,0} + \sqrt{\frac{6}{5}} \text{Im} w_{2,1} - \sqrt{\frac{3}{10}} \text{Re } w_{2,2}$



$\pi\pi$ Scattering and ρ Properties



- fit $\tan(\delta_1) = \frac{\Gamma/2}{m_r - E} + A$ and $\Gamma = \frac{g^2}{48\pi m_r^2}(m_r^2 - 4m_\pi^2)^{3/2}$

Lattice Parameters

- $(32^3|240)$: 412 configs $32^3 \times 256$, $m_\pi \approx 240$ MeV, $m_\pi L \sim 4.4$
- QCD coupling $\beta = 1.5$ such that $a_s \sim 0.12$ fm, $a_t \sim 0.035$ fm
- strange quark mass $m_s = -0.0743$ nearly physical (using kaon)
- work in $m_u = m_d$ limit so $SU(2)$ isospin exact

Collaborators

Dr. Colin Morningstar
Dr. John Bulava
Dr. Brendan Fahy
Andrew Hanlon
Ben Hoerz

Thank you for your time!