

Excited-State Energies and Scattering Phase-Shifts from Lattice QCD

Jake Fallica

Time-separated Euclidean Correlator as Path-Integral on Lattice:

$$\langle \hat{O}^\dagger(t) \hat{O}(0) \rangle = \int [d\Phi] e^{-S[\Phi]} O[\Phi, t] O[\Phi, 0]$$

Time-separated Euclidean Correlator as Sum over Stationary-States:

$$\langle \hat{O}^\dagger(t) \hat{O}(0) \rangle = \sum_n |\langle 0 | \hat{O} | n \rangle|^2 e^{-E_n t}$$

- In principle, could find all energies... Not in practice.



Define Matrix of Correlators!

$$\begin{aligned} C_{ij}(t) &= \left\langle \hat{O}_i^\dagger(t) \hat{O}_j(0) \right\rangle \\ &= \sum_n \langle 0 | \hat{O}_i | n \rangle \langle n | \hat{O}_j^\dagger | 0 \rangle e^{-E_n t} \end{aligned}$$

Diagonalize to pick out Excited-State Energies:

$$\tilde{C}_{ij}(t) = U_\tau^\dagger C_{ij}(t) U_\tau$$

- U diagonalizes C at $t = \tau$
- Choose τ such that C remains diagonal (approx.) for $t > \tau$



Correlator Eigenvalues:

$$\tilde{C}_{nn}(t) = A_n e^{-E_n t} + \dots$$

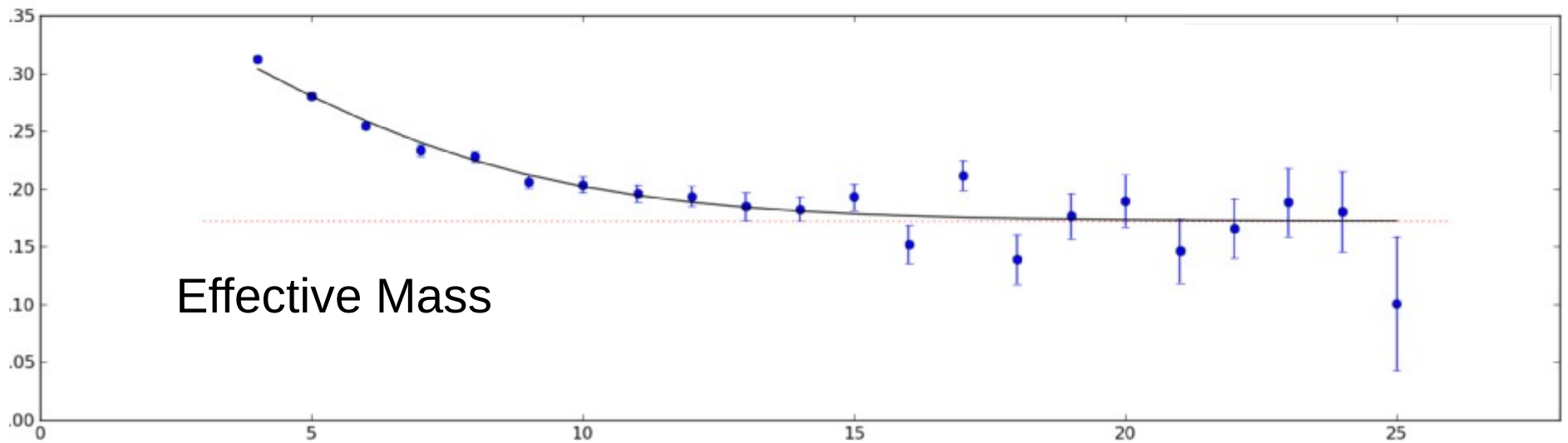
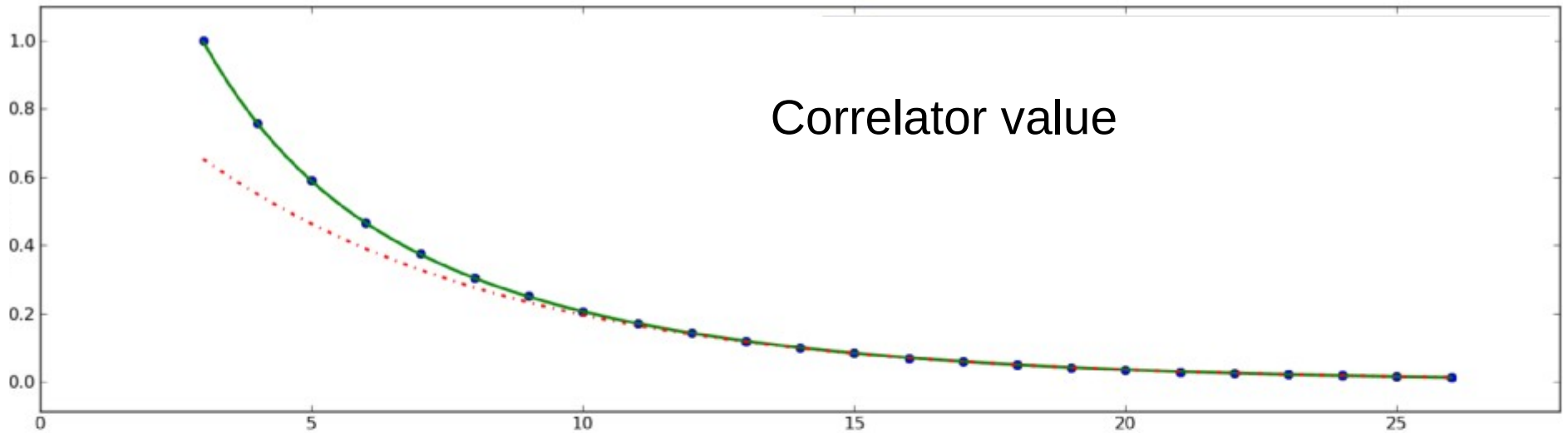
- Sum of decaying exponentials
- Only leading term contributes for large $t \rightarrow$ single exponential fit ?
- Capture excited state contamination \rightarrow double exponential fit !

Effective Mass:

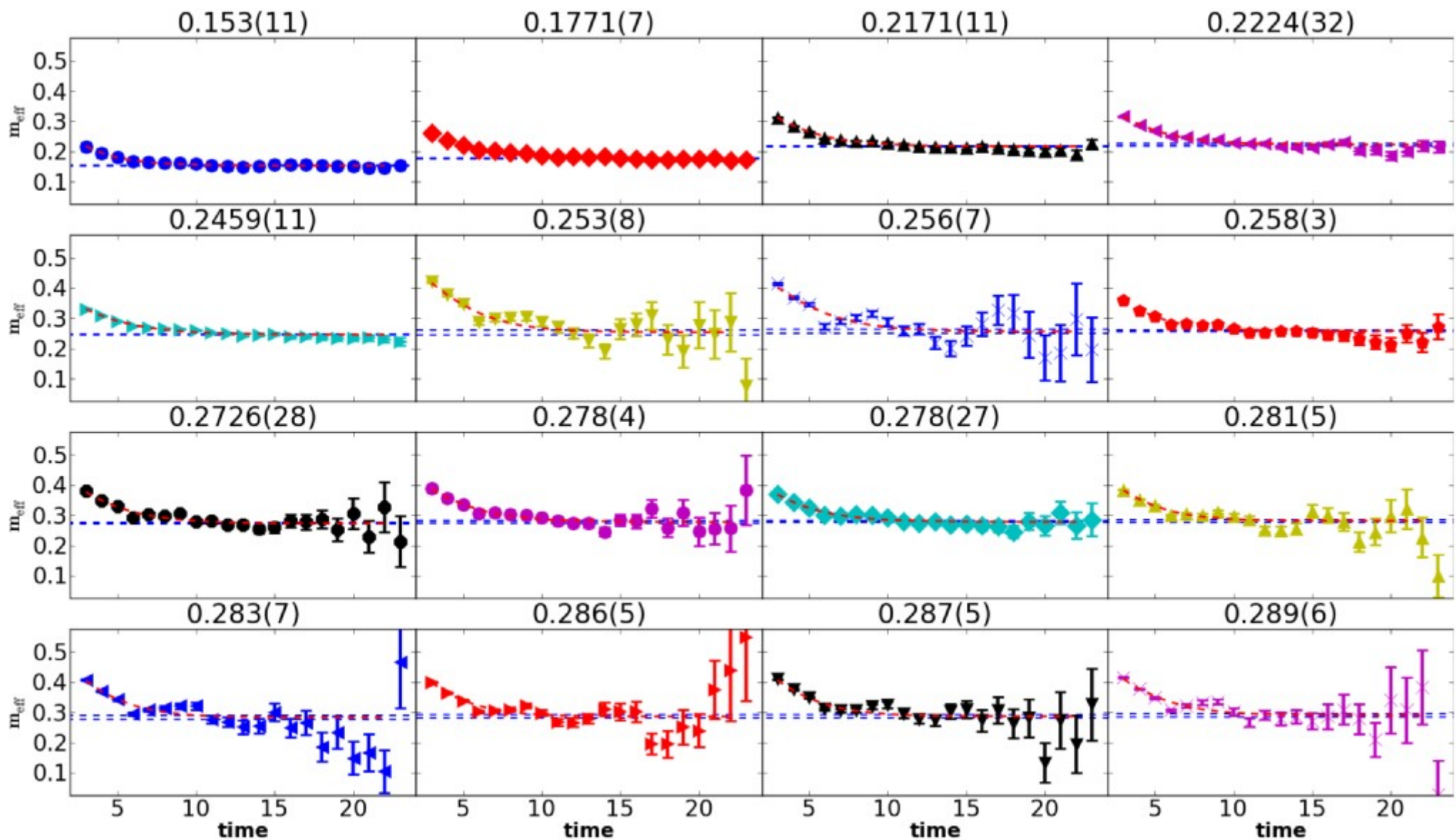
$$m_n^{\text{eff}}(t) = \frac{1}{\Delta t} \ln \left(\frac{\tilde{C}_{nn}(t)}{\tilde{C}_{nn}(t + \Delta t)} \right)$$



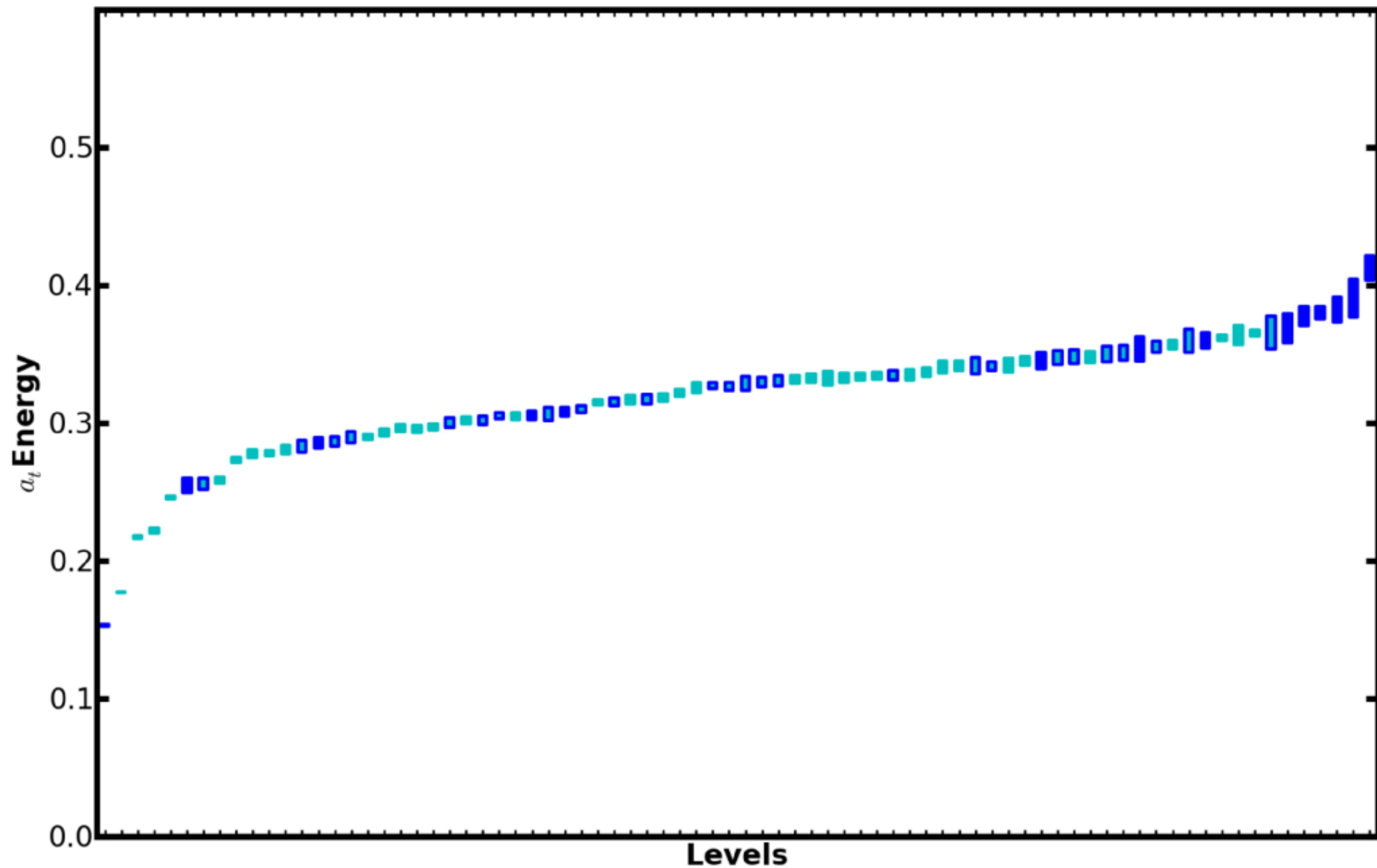
Sample Correlator from Isodoublet Strange T_{1u} :



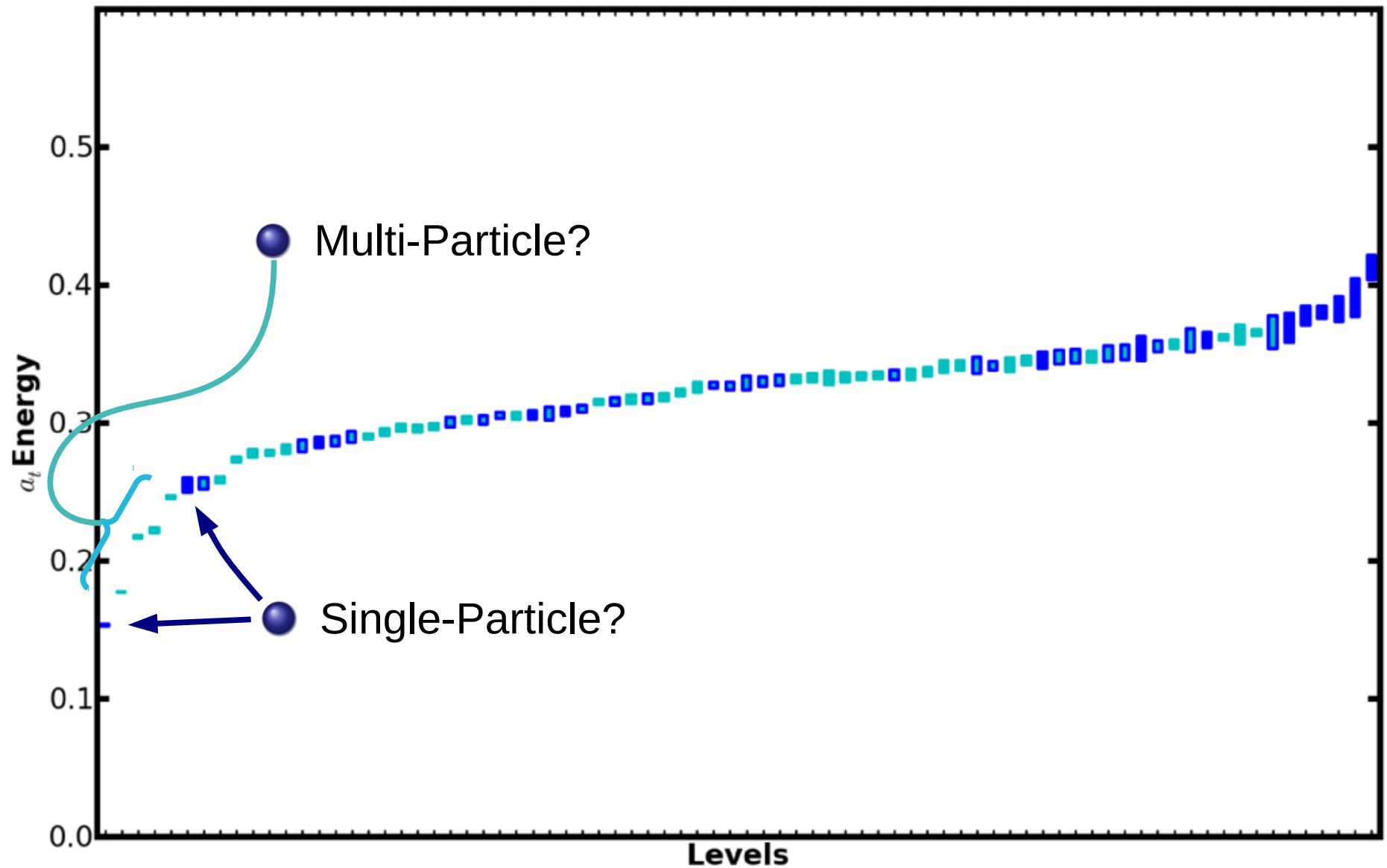
More Effective Masses from Kaon Channel:



Stationary-State Energies in Kaon Channel:



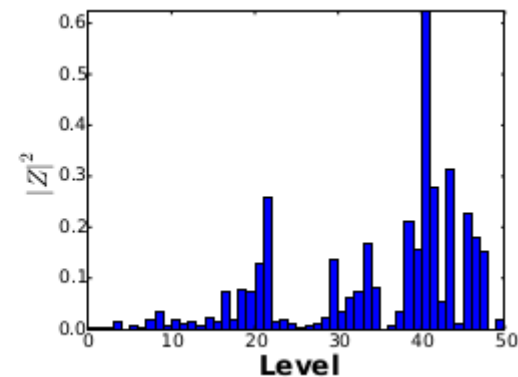
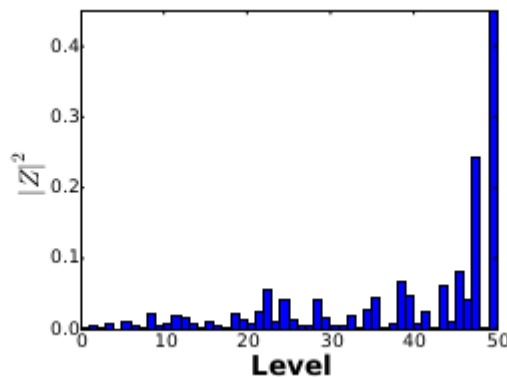
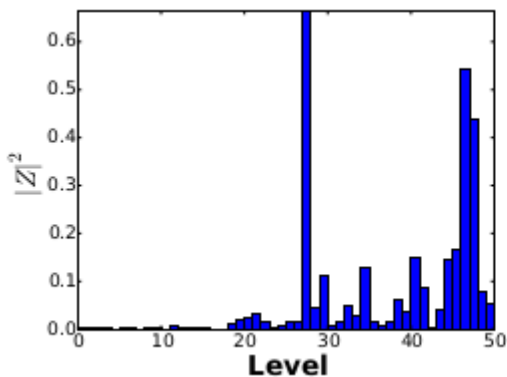
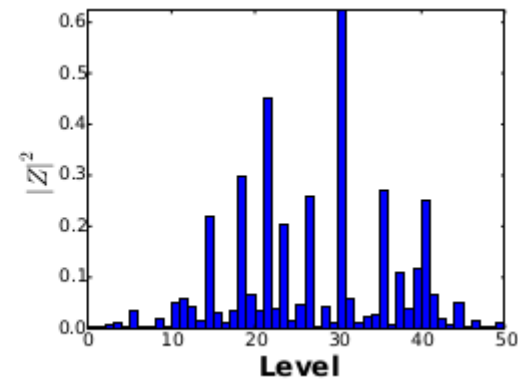
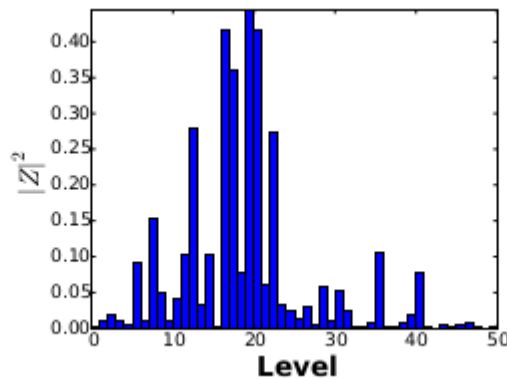
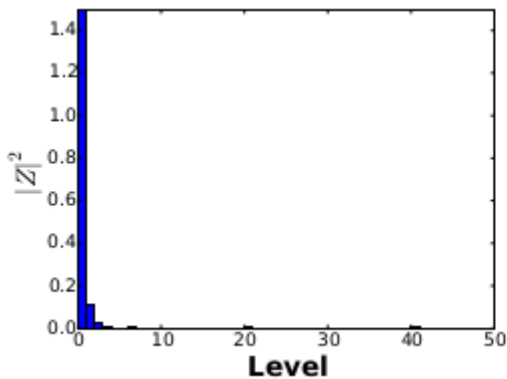
Stationary-State Energies in Kaon Channel:



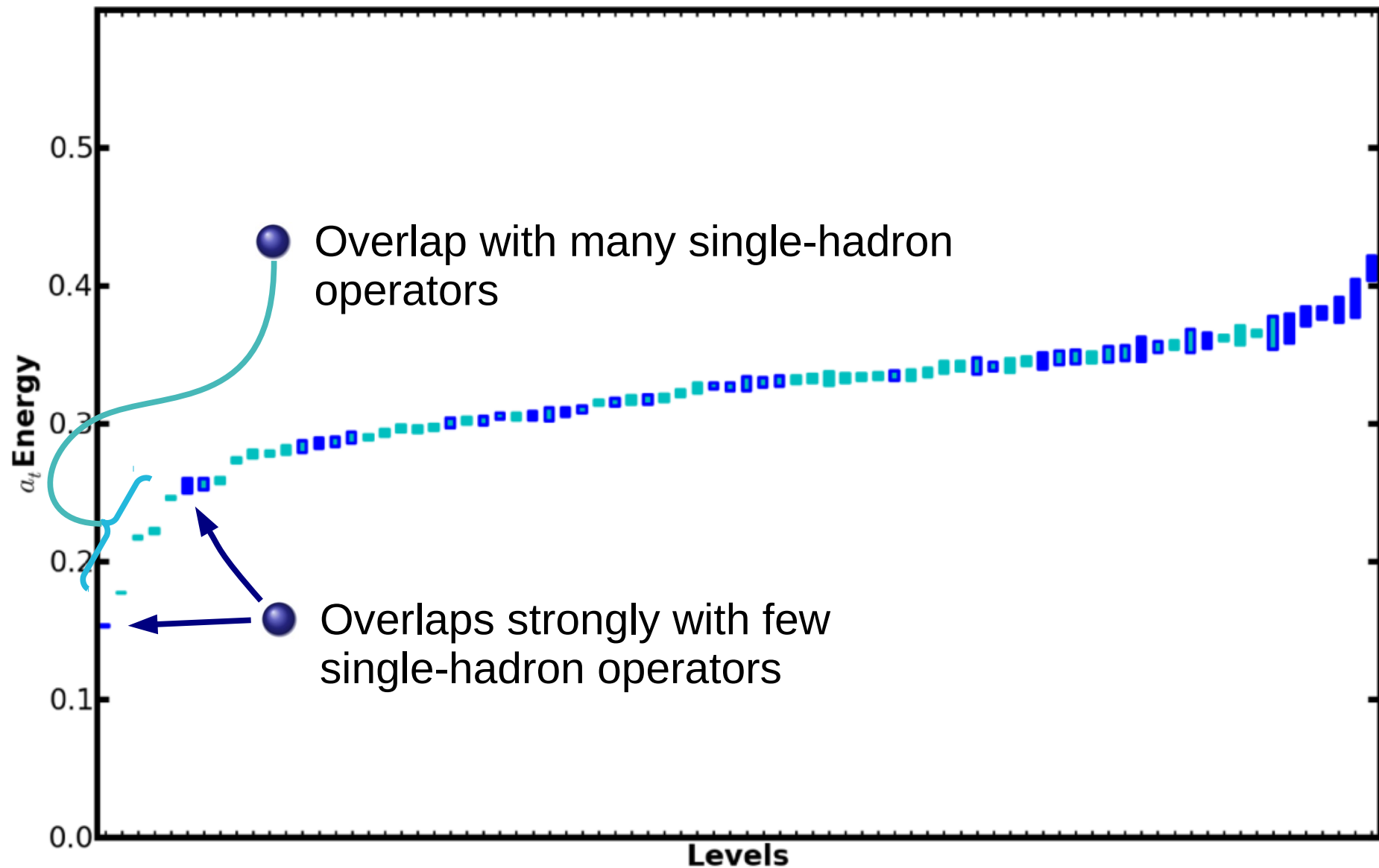
Identifying qqq , $q\bar{q}$ resonances

- Rebuild correlator matrix using ONLY single-hadron operators
- Diagonalize again to find appropriate linear combinations

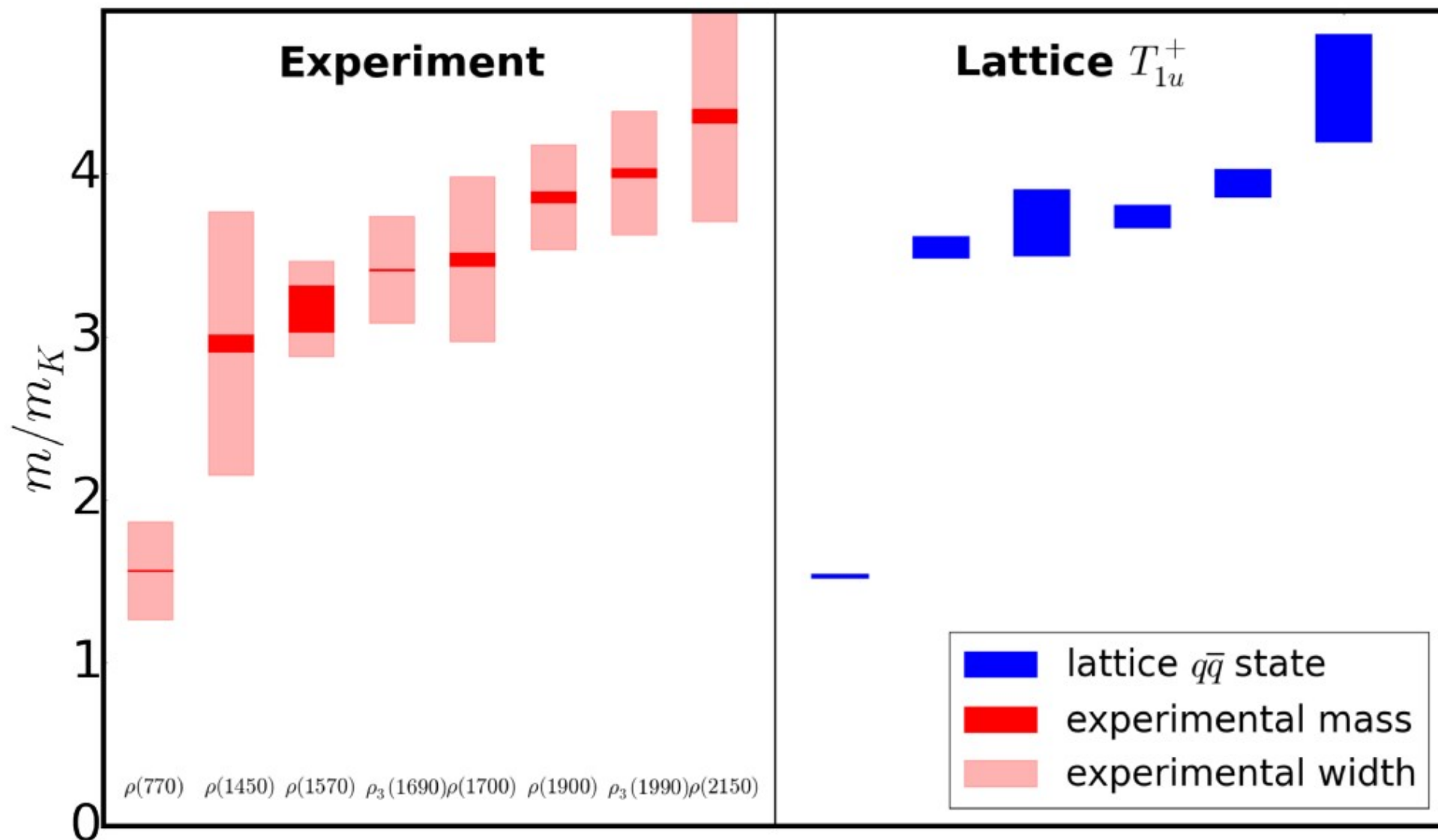
$$\hat{O}'_i{}^{\text{SH}} = \sum U'_{ij} \hat{O}_j^{\text{SH}} \quad \langle n | \hat{O}'_i{}^{\text{SH}\dagger} | 0 \rangle = Z_i^{(n)}$$



Stationary-State Energies in Kaon Channel:



Stationary-State Energies in Rho Channel:



Scattering Phase-Shifts

- correlator of two-particle operator σ in finite volume

$$C^L(P) = \begin{array}{c} \text{Diagram 1} + \text{Diagram 2} \\ + \text{Diagram 3} + \dots \end{array}$$

The diagram shows a series of terms for the correlator $C^L(P)$. Each term consists of two external vertices, σ and σ^\dagger , connected by a chain of internal vertices. The first term is a direct connection. The second term includes one internal vertex labeled iK . The third term includes two iK vertices. Each internal vertex is connected to the chain by two lines, one ending in a black dot and the other in a blue dot. Dashed boxes enclose the internal vertices in each term.

- Bethe-Salpeter kernel

$$\text{Diagram } iK = \begin{array}{c} \text{Diagram 1} + \text{Diagram 2} \\ + \text{Diagram 3} + \text{Diagram 4} \end{array}$$

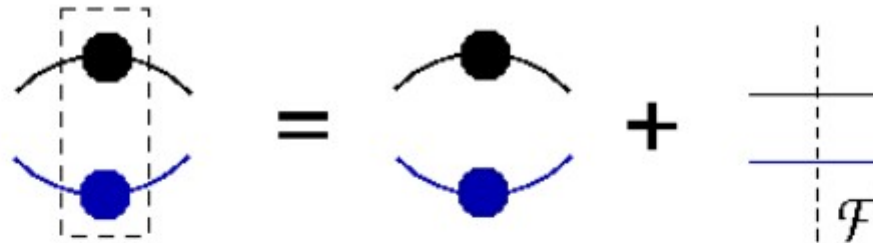
The diagram shows the decomposition of the Bethe-Salpeter kernel iK . It is equal to the sum of four diagrams: 1) a crossed line, 2) a loop with a black dot and a blue dot, 3) a loop with a blue dot and a black dot, and 4) a loop with a green dot.

- $C^\infty(P)$ has branch cuts where two-particle thresholds begin
- momentum quantization in finite volume: cuts \rightarrow series of poles
- C^L poles: two-particle energy spectrum of finite volume theory

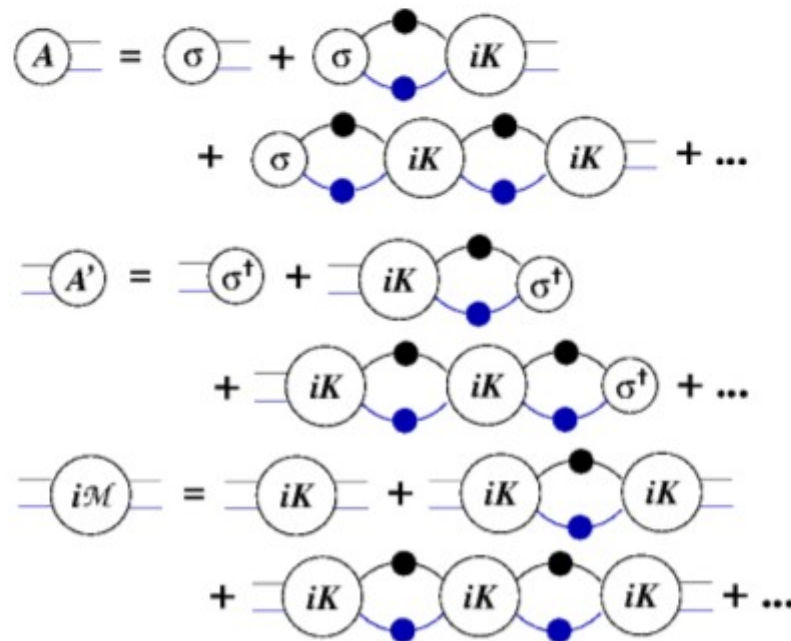


Finite-Volume as Infinite-Volume + Correction

- finite-volume momentum sum is infinite-volume integral plus correction \mathcal{F}



- define the following quantities: A , A' , invariant scattering amplitude $i\mathcal{M}$



Difference Inherits Finite-Volume Poles

- subtracted correlator $C_{\text{sub}}(P) = C^L(P) - C^\infty(P)$ given by

$$C_{\text{sub}}(P) = \begin{array}{c} \textcircled{A} \text{---} \mathcal{F} \text{---} \textcircled{A'} + \textcircled{A} \text{---} \mathcal{F} \text{---} \textcircled{iM} \text{---} \mathcal{F} \text{---} \textcircled{A'} \\ + \textcircled{A} \text{---} \mathcal{F} \text{---} \textcircled{iM} \text{---} \mathcal{F} \text{---} \textcircled{iM} \text{---} \mathcal{F} \text{---} \textcircled{A'} + \dots \end{array}$$

- sum geometric series

$$C_{\text{sub}}(P) = A \mathcal{F} (1 - iM\mathcal{F})^{-1} A'.$$

- poles of $C_{\text{sub}}(P)$ are poles of $C^L(P)$ from $\det(1 - iM\mathcal{F}) = 0$
- E related to S matrix (and phase shifts) by

$$\det[1 + F^{(\mathbf{s}, \gamma, u)}(S - 1)] = 0,$$

where F matrix defined next slide



Too Many Functions

- F matrix in JLS basis states given by

$$F_{J'm_{J'}L'S'a'; Jm_JLSa}^{(\mathbf{s}, \gamma, u)} = \frac{\rho_a}{2} \delta_{a'a} \delta_{S'S} \left\{ \delta_{J'J} \delta_{m_{J'}m_J} \delta_{L'L} \right. \\ \left. + W_{L'm_{L'}; Lm_L}^{(\mathbf{s}, \gamma, u)} \langle J'm_{J'} | L'm_{L'}, Sm_S \rangle \langle Lm_L, Sm_S | Jm_J \rangle \right\}$$

$$W_{L'm_{L'}; Lm_L}^{(\mathbf{s}, \gamma, u)} = \frac{2i}{\pi \gamma u^{l+1}} \mathcal{Z}_{lm}(\mathbf{s}, \gamma, u^2) \int d^2\Omega Y_{L'm_{L'}}^*(\Omega) Y_{lm}^*(\Omega) Y_{Lm_L}(\Omega)$$

$$\mathcal{Z}_{lm}(\mathbf{s}, \gamma, u^2) = \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{\mathcal{Y}_{lm}(\mathbf{z})}{(z^2 - u^2)} e^{-\Lambda(z^2 - u^2)}$$

$$+ \delta_{l0} \gamma \pi e^{\Lambda u^2} \left(2u D(u\sqrt{\Lambda}) - \Lambda^{-1/2} \right)$$

$$+ \frac{i^l \gamma}{\Lambda^{l+1/2}} \int_0^1 dt \left(\frac{\pi}{t} \right)^{l+3/2} e^{\Lambda t u^2} \sum_{\substack{\mathbf{n} \in \mathbb{Z}^3 \\ \mathbf{n} \neq 0}} e^{\pi i \mathbf{n} \cdot \mathbf{s}} \mathcal{Y}_{lm}(\mathbf{w}) e^{-\pi^2 \mathbf{w}^2 / (t\Lambda)}$$



Too Many Functions (cont.)

$$\mathbf{z} = \mathbf{n} - \gamma^{-1} \left[\frac{1}{2} + (\gamma - 1) s^{-2} \mathbf{n} \cdot \mathbf{s} \right] \mathbf{s},$$

$$\mathbf{w} = \mathbf{n} - (1 - \gamma) s^{-2} \mathbf{s} \cdot \mathbf{n} \mathbf{s}, \quad \mathcal{Y}_{lm}(\mathbf{x}) = |\mathbf{x}|^l Y_{lm}(\hat{\mathbf{x}})$$

$$D(x) = e^{-x^2} \int_0^x dt e^{t^2} \quad (\text{Dawson function})$$

$$E_{\text{cm}} = \sqrt{E^2 - \mathbf{P}^2}, \quad \gamma = \frac{E}{E_{\text{cm}}},$$

$$\mathbf{q}_{\text{cm}}^2 = \frac{1}{4} E_{\text{cm}}^2 - \frac{1}{2} (m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4E_{\text{cm}}^2},$$

$$u^2 = \frac{L^2 \mathbf{q}_{\text{cm}}^2}{(2\pi)^2}, \quad \mathbf{s} = \left(1 + \frac{(m_1^2 - m_2^2)}{E_{\text{cm}}^2} \right) \mathbf{d}$$



Phase-Shifts in terms of Energies:

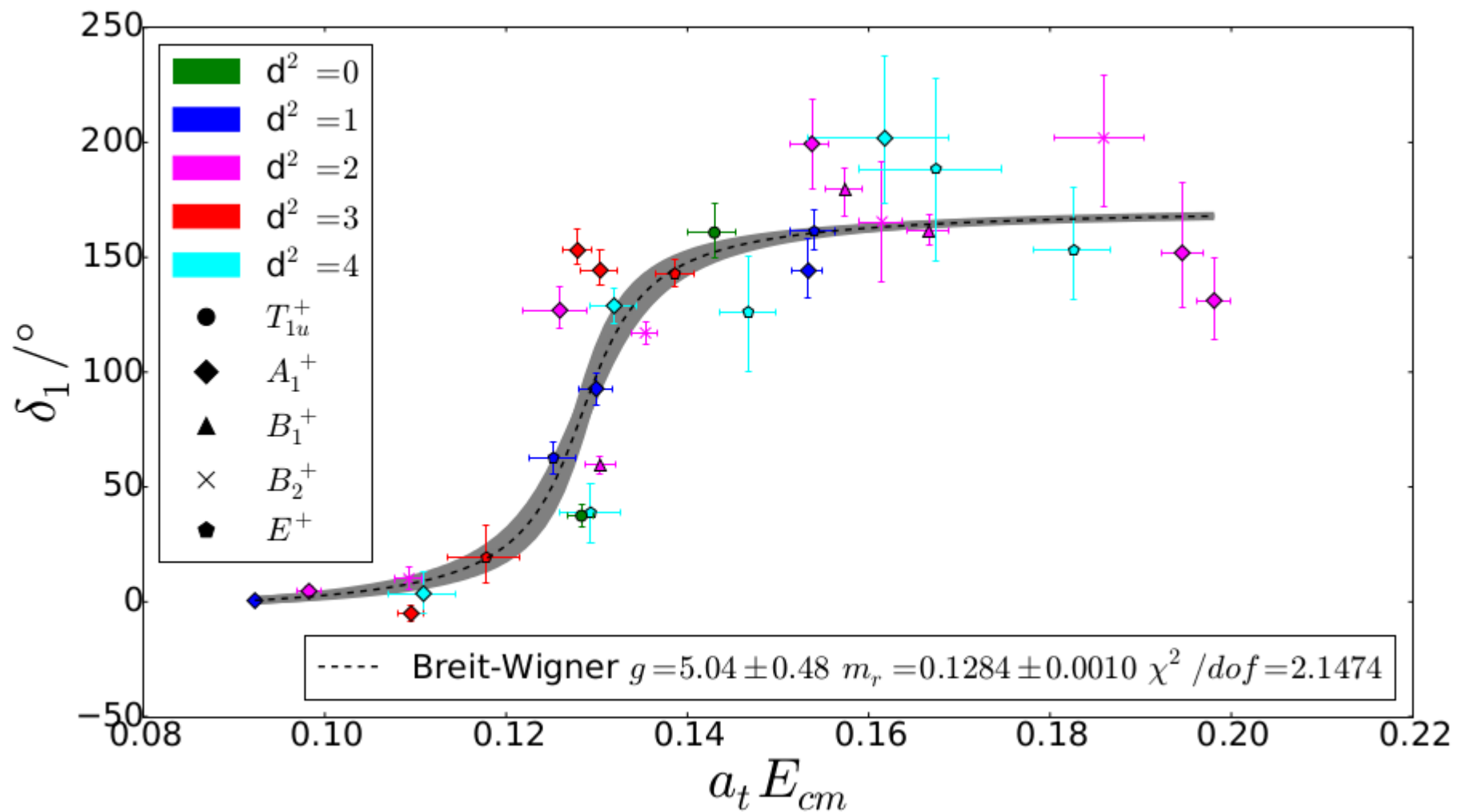
- for P -wave phase shift $\delta_1(E_{\text{cm}})$ for $\pi\pi I = 1$ scattering
- define

$$w_{lm} = \frac{\mathcal{Z}_{lm}(s, \gamma, u^2)}{\gamma\pi^{3/2}u^{l+1}}$$

d	Λ	$\cot \delta_1$
(0,0,0)	T_{1u}^+	$\text{Re } w_{0,0}$
(0,0,1)	A_1^+	$\text{Re } w_{0,0} + \frac{2}{\sqrt{5}} \text{Re } w_{2,0}$
	E^+	$\text{Re } w_{0,0} - \frac{1}{\sqrt{5}} \text{Re } w_{2,0}$
(0,1,1)	A_1^+	$\text{Re } w_{0,0} + \frac{1}{2\sqrt{5}} \text{Re } w_{2,0} - \sqrt{\frac{6}{5}} \text{Im } w_{2,1} - \sqrt{\frac{3}{10}} \text{Re } w_{2,2},$
	B_1^+	$\text{Re } w_{0,0} - \frac{1}{\sqrt{5}} \text{Re } w_{2,0} + \sqrt{\frac{6}{5}} \text{Re } w_{2,2},$
	B_2^+	$\text{Re } w_{0,0} + \frac{1}{2\sqrt{5}} \text{Re } w_{2,0} + \sqrt{\frac{6}{5}} \text{Im } w_{2,1} - \sqrt{\frac{3}{10}} \text{Re } w_{2,2}$



$\pi\pi$ Scattering and ρ Properties



● fit $\tan(\delta_1) = \frac{\Gamma/2}{m_r - E} + A$ and $\Gamma = \frac{g^2}{48\pi m_r^2} (m_r^2 - 4m_\pi^2)^{3/2}$



Lattice Parameters

- $(32^3|240)$: 412 configs $32^3 \times 256$, $m_\pi \approx 240$ MeV, $m_\pi L \sim 4.4$
- QCD coupling $\beta = 1.5$ such that $a_s \sim 0.12$ fm, $a_t \sim 0.035$ fm
- strange quark mass $m_s = -0.0743$ nearly physical (using kaon)
- work in $m_u = m_d$ limit so $SU(2)$ isospin exact

Collaborators

Dr. Colin Morningstar
Dr. John Bulava
Dr. Brendan Fahy
Andrew Hanlon
Ben Hoerz

Thank you for your time!